Connection Dynamics for Higher Dimensional Scalar-Tensor Theories of Gravity

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The scalar-tensor theories of gravity in spacetime dimensions $D+1>2$ are studied. By doing Hamiltonian analysis, we obtain the geometrical dynamics of the theories from their Lagrangian. The Hamiltonian formalism indicates that the theories are naturally divided into two sectors by the coupling parameter ω. The Hamiltonian structure in both sectors are similar to the corresponding structure of 4-dimensional cases. It turns out that there is a symplectic reduction from the canonical structure of $so(D+1)$ Yang-Mills theories coupled to the scalar field to the canonical structure of the geometrical scalar-tensor theories. Therefore the non-perturbative loop quantum gravity techniques can also be applied to the scalar-tensor theories in $D+1$ dimensions based on their connection-dynamical formalism.

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I. INTRODUCTION

Ever since 1998, a few independent astronomic observations strongly suggested that our Universe is currently undergoing a period of acceleration [1]. This causes the “dark energy” problem in the framework of general relativity (GR). While a positive cosmological constant Λ could be employed to explain the acceleration, the observed value of Λ is unexpectedly much smaller than any theoretical estimation. Therefore it is reasonable to consider the possiblity that GR is not a valid theory of gravity on galactic or cosmological scale. For this reason, as well as some non-trivial tests on gravity beyond GR [2–4], modified gravity theories have received increased attention recently. Among various alternative models, a simple and typical modification of GR is the so-called $f(\mathcal{R})$ theories [5]. Besides $f(\mathcal{R})$ theories, Brans-Dicke theory of gravity which was first proposed by Brans and Dicke in 1961 and compatible with Mach's principle [6] also caught much attention. In this theory, a scalar field representing the varying “gravitational constant” is non-minimally coupled to the scalar curvature. To interpret the observational results within the framework of a broad class of theories, the Brans-Dicke theory was generalized by Bergmann [7] and Wagoner [8] to scalar-tensor theories (STT). The scalar field in STT of gravity is expected to account for the mysterious “dark energy”, since it can naturally lead to cosmological acceleration in certain models (see e. g. [9–11]). In particular, the current acceleration of the Universe can be naturally obtained in 5-dimensional Brans-Dicke theory without a fine-tuning of the coupling parameter [12]. Moreover, some models of STT of gravity may also account for the “dark matter” problem [13–15], which was revealed by the observed rotation curve of galaxy clusters. Besides, scalar-tensor modifications of GR have also become very popular as the low-energy and effective limit in unification schemes such as bosonic string theory (see e. g. [16–18]). It should be noted that the general scalar-tensor theory can include both metric $f(\mathcal{R})$ theories and Palatini $f(\mathcal{R})$ theories as special sectors with different coupling parameter ω, while the original Brans-Dicke theory is the particular case of constant ω and vanishing potential.

On the other hand, during the past several decades, seeking for a quantum theory of gravity has been a rather active field. Among various kinds of programmes, loop quantum gravity(LQG), a background independent approach to quantize general relativity, has widely been investigated [19–22]. Surprisingly, as a non-renormalizable theory from the view of perturbative quantum field theory, GR can be non-perturbatively quantized by the loop quantization procedure. The loop quantization programme heavily relies on the connection-dynamical formulation of GR, which requires a Poission self-commuting connection variable and a corresponding compact gauge group. While the approach to formulate the connection dynamics was originally restricted to 4-dimensional GR, it can also be generalized to 4-dimensional $f(R)$ theories and general STT [23–27]. However, modern theoretical research indicates that we might live in a universe with spacetime dimension $D+1>4$. Thus one is naturally led to ask whether higher dimensional gravity theories can be formulated as gauge theories with connection dynamics. Recently, in a series of seminal articles [28, 29], Bodendorfer, Thiemann and Thurn successfully developed an approach to formulate the connection

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dynamics for GR as well as supergravity theories in higher dimensions [30, 31]. Taking account of the cosmological and astrophysical significance, it is desirable to study if the connection-dynamical formalism also exists for STT in arbitrary dimensions. In this paper we will give an affirmative answer to this question. Our results can serve as the starting-point for the non-perturbative loop quantization of STT in higher dimensions.

This paper is organized as follows. In section 2, we formulate the Hamiltonian analysis of $(D+1)$-dimensional $(D>1)$ STT in terms of ADM variables. In section 3, we first give a brief review of the new variables and connection dynamics of GR in $D+1$ dimensions. Then we show how to obtain the ADM variables from the new connection variables of $(D+1)$-dimensional STT by symplectic reduction. We will write out the explicit form of the four different constraints and prove that they indeed form a first-class constraint system when $\omega(\phi) \neq -\frac{D}{D-1}$, For the special case when $\omega(\phi) = -\frac{D}{D-1}$, a new constraint generating spacetime conformal transformations is found. The five different constraints also form a first-class system. We summarize our results in the last section. The detailed calculations of several Poisson brackets will be given in appendix A . Throughout the paper, we use Greek alphabet $\mu, \nu, ...$ for spacetime indices, Latin alphabet $a, b, c, ...,$ for spatial indices, and $I, J, K, ...,$ for internal indices.

II. HAMILTONIAN ANALYSIS

The general action of scalar-tensor theories in $D+1$ dimensions reads

$$S[g,\phi] = \int_\Sigma d^{D+1}x\sqrt{-g}\Big[\frac{1}{2}\Big(\phi\mathcal{R} - \frac{\omega(\phi)}{\phi}(\partial_\mu\phi)\partial^\mu\phi\Big) - \xi(\phi)\Big], \tag{2.1}$$

where we set $8\pi G = 1$, $g \equiv det(g_{\mu\nu})$, $\mathcal{R}$ denotes the scalar curvature of spacetime metric $g_{\mu\nu}$, the coupling parameter $\omega(\phi)$ and potential $\xi(\phi)$ can be arbitrary functions of scalar field ϕ. Variations of the action (2.1) with respect to $g_{\mu\nu}$ and ϕ give equations of motion:

$$\phi G_{\mu\nu} = \nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi + \frac{\omega(\phi)}{\phi}[(\partial_\mu\phi)\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2] - g_{\mu\nu}\xi(\phi), \tag{2.2}$$

$$\mathcal{R} = -\frac{2\omega(\phi)}{\phi}\Box\phi + \frac{\omega(\phi)}{\phi^2}(\partial_\mu\phi)\partial^\mu\phi - \frac{\omega'(\phi)}{\phi}(\partial_\mu\phi)\partial^\mu\phi + 2\xi'(\phi), \tag{2.3}$$

where a prime over the letter denotes a derivative with respect to the argument, ∇_μ is the covariant derivative compatible with $g_{\mu\nu}$ and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. By doing D+1 decomposition of the spacetime, the $(D+1)$-dimensional scalar curvature can be expressed as

$$\mathcal{R} = K_{ab}K^{ab} - K^2 + R^{(D)} + \frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g}n^\mu K) - \frac{2}{N\sqrt{h}}\partial_a(\sqrt{h}h^{ab}\partial_b N), \tag{2.4}$$

where K_{ab} is the extrinsic curvature of a spatial hypersurface Σ, $K \equiv K_{ab}h^{ab}$, $h \equiv det(h_{ab})$, $R^{(D)}$ denotes the scalar curvature of the D-metric h_{ab} induced on Σ, n^μ is the unit normal of Σ and N is the lapse function. By Legendre transformation, the momenta conjugate to the dynamical variables h_{ab} and ϕ are defined respectively as

$$p^{ab} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ab}} = \frac{\sqrt{h}}{2}[\phi(K^{ab} - Kh^{ab}) - \frac{h^{ab}}{N}(\dot{\phi} - N^c\partial_c\phi)], \tag{2.5}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = -\sqrt{h}(K - \frac{\omega(\phi)}{N\phi}(\dot{\phi} - N^c\partial_c\phi)), \tag{2.6}$$

where N^c is the shift vector. The combination of the trace of Eq. (2.5) and Eq. (2.6) gives

$$(D + (D-1)\omega(\phi))(\dot{\phi} - N^a\partial_a\phi) = \frac{2N}{\sqrt{h}}(\frac{D-1}{2}\phi\pi - p), \tag{2.7}$$

where $p \equiv p_c{}^c$ is the trace of p^{ab}. It is easy to see from Eq. (2.7) that one extra constraint $C = p - \frac{D-1}{2}\phi\pi = 0$ emerges when $\omega(\phi) = -\frac{D}{D-1}$. So it is natural to mark off two sectors of the theories by $\omega(\phi) \neq -\frac{D}{D-1}$ and $\omega(\phi) = -\frac{D}{D-1}$.

A. Sector of $\omega(\phi) \neq -\frac{D}{D-1}$

In the case of $\omega(\phi) \neq -\frac{D}{D-1}$, the Hamiltonian of STT can be derived as a liner combination of constraints as

$$H_{total} = \int_\Sigma d^Dx(N^aH_a + NH), \tag{2.8}$$

where the smeared diffeomorphism and Hamiltonian constraints read respectively

$$H[\overrightarrow{N}] = \int_\Sigma d^DxN^aH_a = \int_\Sigma d^DxN^a\left(-2D^b(p_{ab})+\pi\partial_a\phi\right), \tag{2.9}$$

$$\begin{aligned} H[N] &= \int_\Sigma d^DxNH \\ &= \int_\Sigma d^DxN\Big[\frac{2}{\sqrt{h}}\left(\frac{p_{ab}p^{ab}-\frac{1}{D-1}p^2}{\phi}+\frac{(p-\frac{D-1}{2}\phi\pi)^2}{\phi(D-1)(D+(D-1)\omega)}\right) \\ &+ \frac{1}{2}\sqrt{h}(-\phi R^{(D)}+\frac{\omega(\phi)}{\phi}(D_a\phi)D^a\phi+2D_aD^a\phi+2\xi(\phi))\Big], \end{aligned} \tag{2.10}$$

where D_a denote the spatial covariant derivative compatible with h_{ab}. By the symplectic structure

$$\begin{aligned} \{h_{ab}(x),p^{cd}(y)\} &= \delta_a^{(c}\delta_b^{d)}\delta^D(x,y), \\ \{\phi(x),\pi(y)\} &= \delta^D(x,y), \end{aligned} \tag{2.11}$$

lengthy but straightforward calculations show that the constraints (2.9) and (2.10) comprise a first-class system similar to GR as:

$$\begin{aligned} \{H[\overrightarrow{N}],H[\overrightarrow{N}']\} &= H([\overrightarrow{N},\overrightarrow{N}']), \\ \{H[M],H[\overrightarrow{N}]\} &= H[-\mathcal{L}_{\overrightarrow{N}}M], \\ \{H[N],H[M]\} &= H_a[ND^aM-MD^aN]. \end{aligned} \tag{2.12}$$

We now show that the above Hamiltonian formalism of STT in D dimensions is equivalent to their initial value formalism. In the Hamiltonian framework, the evolution equations of the basic canonical variables can be derived by taking Poisson brackets with the Hamiltonian (2.8). Thus it is easy to check that the evolution equation of h_{ab} is just the definition of K_{ab}. Also we have

$$\dot\phi = \{\phi, H_{total}\} = \frac{2N}{(D+(D-1)\omega)\sqrt{h}}(\frac{D-1}{2}\phi\pi-p)+N^a\partial_a\phi, \tag{2.13}$$

which is nothing but Eq. (2.7). Moreover, it is easy to obtain

$$\begin{aligned} \dot\pi &= \partial_a(N^a\pi)+\frac{N\sqrt{h}}{2}(K_{ab}K^{ab}-K^2)+\frac{N\sqrt{h}}{2}R-\partial_a(\sqrt{h}h^{ab}\partial_bN) \\ &+ \frac{N\omega\sqrt{h}}{2\phi^2}(D_a\phi)D^a\phi+\sqrt{h}\omega D_a(\frac{N}{\phi}D^a\phi)-\frac{\omega\sqrt{h}}{2\phi^2N}(\dot\phi-N^c\partial_c\phi)^2+\frac{\omega'(\phi)N\sqrt{h}}{2\phi}(D_a\phi)D^a\phi-N\sqrt{h}\xi'(\phi). \end{aligned} \tag{2.14}$$

Using Eqs. (2.4), (2.6) and $n^0=\frac{1}{N}, n^a=-\frac{N^a}{N}, \sqrt{-g}=N\sqrt{h}$, we can prove that Eq.(2.14) is equivalent to Eq.(2.3). The equation of motion for p_{ab} reads

$$\begin{aligned} \dot p_{ab} &= \frac{h_{ab}N}{\sqrt{h}}(\frac{p_{cd}p^{cd}-\frac{1}{D-1}p^2}{\phi}+\frac{(p-\frac{D-1}{2}\phi\pi)^2}{(D+1)\phi(D+(D-1)\omega)})+\frac{4N}{\sqrt{h}}(\frac{p_{ac}p_b{}^c-\frac{1}{D-1}pp_{ab}}{\phi}+\frac{(p-\frac{D-1}{2}\phi\pi)p_{ab}}{(D-1)\phi(D+(D-1)\omega)}) \\ &+ \frac{N}{4}\sqrt{h}h_{ab}\phi R^{(D)}-\frac{N}{2}\sqrt{h}\phi R_{ab}-\frac{N}{2}\sqrt{h}h_{ab}D_cD^c\phi-D_{(a}N\sqrt{h}D_{b)}\phi-\frac{N\omega}{4\phi}\sqrt{h}h_{ab}(D_c\phi)D^c\phi+\frac{N\omega}{2\phi}\sqrt{h}(D_a\phi)D_b\phi \\ &+ \frac{\sqrt{h}}{2}(D_{(a}D_{b)}(N\phi)-h_{ab}D_cD^c(N\phi))+2p_{c(a}D^cN_{b)}+D_c(p_{ab}N^c)-\frac{1}{2}N\sqrt{h}h_{ab}\xi(\phi). \end{aligned} \tag{2.15}$$

We can use Eq. (2.15) to derive the time derivative of the extrinsic curvature:

$$K_{ab} = \frac{2(p_{ab}-\frac{1+\omega}{D+(D-1)\omega}ph_{ab})}{\phi\sqrt{h}}-\frac{\pi h_{ab}}{(D+(D-1)\omega)\sqrt{h}}. \tag{2.16}$$

Straightforward calculations give

$$\begin{aligned} \dot K_{ab} &= 2NK_{ac}K_b^c-NKK_{ab}+\mathcal{L}_{\overrightarrow{N}}K_{ab}-NR_{ab}+D_aD_bN+\frac{N}{\phi}D_aD_b\phi \\ &+ \frac{N\omega}{\phi^2}(D_a\phi)D_b\phi-\frac{n^\sigma\partial_\sigma\phi}{\phi}NK_{ab}+\frac{Nh_{ab}}{\phi}(\frac{1}{D-1}\Box\phi+\frac{2}{D-1}\xi(\phi)). \end{aligned} \tag{2.17}$$

It is not difficult to see that Eq. (2.17) is in accordance with Eq. (2.2). Thus we have proved that the Hamiltonian formalism of STT with is equivalent to their Lagrangian formalism when $\omega(\phi) \neq -\frac{D}{D-1}$.

B. Sector of $\omega(\phi) = -\frac{D}{D-1}$

In the special case of $\omega(\phi) = -\frac{D}{D-1}$, Eq. (2.7) implied an extra primary constraint $C = 0$, which we call "conformal constraint". Hence, the total Hamiltonian now can be expressed as a liner combination of constraints as

$$H_{total} = \int_\Sigma d^D x(N^a H_a + NH + \lambda C), \tag{2.18}$$

where the smeared diffeomorphism $H(\overrightarrow{N})$ is as same as Eq.(2.9), while the Hamiltonian and conformal constraints read respectively:

$$\begin{aligned} H[N] &= \int_\Sigma d^D x NH \\ &= \int_\Sigma d^D x N\left[\frac{2}{\sqrt{h}}\left(\frac{p_{ab}p^{ab} - \frac{1}{D-1}p^2}{\phi}\right) + \frac{1}{2}\sqrt{h}(-\phi R^{(D)} - \frac{D}{(D-1)\phi}(D_a\phi)D^a\phi + 2D_aD^a\phi + 2\xi(\phi))\right] \quad (2.19)\\ C[\lambda] &= \int_\Sigma d^D x \lambda C = \int_\Sigma d^D x\lambda(p - \frac{D-1}{2}\phi\pi). \quad (2.20) \end{aligned}$$

By the symplectic structure (2.11), detailed calculations show that

$$\begin{aligned} \{H[M], H[\overrightarrow{N}]\} &= H[-\mathcal{L}_{\overrightarrow{N}}M], \quad \{C[\lambda], H[\overrightarrow{N}]\} = C[-\mathcal{L}_{\overrightarrow{N}}\lambda], \quad (2.21)\\ \{H[N], H[M]\} &= H_a[ND^aM - MD^aN] + C[\frac{2D_a\phi}{(D-1)\phi}(ND^aM - MD^aN)], \quad (2.22)\\ \{C[\lambda], H[M]\} &= H[\frac{\lambda M}{2}] + \int_\Sigma N\lambda\sqrt{h}(-\frac{D+1}{2}\xi(\phi) + \frac{D-1}{2}\phi\xi'(\phi)). \quad (2.23) \end{aligned}$$

One may understand Eqs. (2.21) by the geometrical interpretations of $H(\overrightarrow{N})$. The Poisson bracket (2.23) implies that, in order to maintain the constraints S and H in the time evolution, we have to impose a secondary constraint

$$-\frac{D+1}{2}\xi(\phi) + \frac{D-1}{2}\phi\xi'(\phi) = 0. \tag{2.24}$$

It is easy to see that this constraint is of second-class and hence has to be solved. As for the vacuum case that we considered the solutions of Eq. (2.24) are of two types,

$$\xi(\phi) = 0 \quad or \quad \xi(\phi) = c\phi^{\frac{D+1}{D-1}}, \tag{2.25}$$

where c is certain dimensional constant. Thus the consistency strongly restricted the feasible STT in the sector $\omega(\phi) = -\frac{D}{D-1}$ to only two theories. For these two theories, the action (2.1) is invariant under the following conformal transformations:

$$g_{\mu\nu} \to e^\lambda g_{\mu\nu}, \quad \phi \to e^{-\frac{D-1}{2}\lambda}\phi. \tag{2.26}$$

For this reason, besides diffeomorphism invariance, the two theories are also conformal invariant. Now in the Hamiltonian formalism the constraints (H, H_a, C) comprise a first-class system. The conformal constraint generates the following transformations on the phase space

$$\{h_{ab}, C(\lambda)\} = \lambda h_{ab}, \quad \{p^{ab}, C(\lambda)\} = -\lambda P^{ab}, \tag{2.27}$$

$$\{\phi, C(\lambda)\} = -\frac{D-1}{2}\lambda\phi, \quad \{\pi, C(\lambda)\} = \frac{D-1}{2}\lambda\pi. \tag{2.28}$$

It is easy to check that the above transformations coincide with those of spacetime conformal transformations (2.26). Thus the physical meaning of the constraints are clear. Because of the extra conformal constraint (2.20), the physical degrees of freedom of this special kind of STT are equal to those of GR in D+1 dimensions. Since the initial value formalism in this sector is a delicate issue [5, 32], we leave the comparison between the Hamiltonian formulation and the Lagrangian formulation for further study.

III. CONNECTION-DYNAMICAL FORMULATION

A. Review of the connection dynamics for GR in $D+1$ dimensions

In this subsection, we will give a brief introduction to the approach in Ref.[28] for constructing the connection dynamics of GR in arbitrary dimensions. The framework will be employed to formulate the connection dynamics of STT in $D+1$ dimensions in the next subsection.

As is well known, the ADM Hamiltonian formulation of vacuum $(D+1)$-dimensional GR is based on a phase space coordinatised by a canonical pair (h_{ab}, P^{ab}) with Poisson brackets

$$\{h_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^D(x,y), \quad \{h_{ab}(x), h_{cd}(y)\} = \{P^{ab}(x), P^{cd}(y)\} = 0. \tag{3.1}$$

The spatial diffeomorphism constraint and Hamiltonian constraint for Lorentzian spacetime read respectively

$$V_a = -2h_{ac}D_bP^{bc}, \tag{3.2}$$

$$H = -\frac{1}{2}\sqrt{det(h)}R^{(D)} + \frac{2}{\sqrt{det(h)}}(h_{ac}h_{bd} - \frac{1}{D-1}h_{ab}h_{cd})P^{ab}P^{cd}. \tag{3.3}$$

To formulate GR in terms of a gauge theory, the central idea is to extend the ADM phase space by additional degrees of freedom and then impose additional first-class constraints such that, after sympletic reduction with respect to these constraints, we can recover the original ADM phase space. The canonical pair of the extended phase space consists of a Lie algebra valued one form A_{aIJ} with dimension N and the corresponding conjugate momentum π^{aIJ} which is a Lie algebra valued weight-one vector density.

It is argued in [28] that the underlying gauge group which one should choose without gauge fixing is $SO(1,D)$ or $SO(D+1)$ for $(D+1)$-dimensional spacetime, and an additional constraint will appear due to the mismatching between the number of the degrees of freedom of (A_{aIJ}, π^{aIJ}) and (h_{ab}, P^{ab}) modulo the Gaussian constraints. In practical terms, the degrees of freedom for ${A_a}^{IJ}$ are $\frac{D^2(D+1)}{2}$ where $\{a \in 1...D\}$ and $\{I, J \in 0...D\}$. Note that the two internal indices of ${A_a}^{IJ}$ are antisymmetric with each other, and hence it contributes $\frac{D(D+1)}{2}$ degrees of freedom. After subtracting the number of Gaussian constraints, $\frac{D(D+1)}{2}$, and the degrees of freedom of h_{ab}, $\frac{D(D+1)}{2}$, the remaining degrees of freedom read $\frac{D^2(D+1)}{2} - \frac{D(D+1)}{2} - \frac{D(D+1)}{2} = \frac{D(D-2)(D+1)}{2}$, which means there are $\frac{D(D-2)(D+1)}{2}$ additional constraints. These constraints could be imposed on the momentum π^{aIJ} conjugate to A_{aIJ}, if we require π^{aIJ} be determined by the co-D-bein e^I_a. Since π^{aIJ} has degrees of freedom $\frac{D^2(D+1)}{2}$, while e^I_a has only $D(D+1)$, the subtraction $\frac{D^2(D+1)}{2} - D(D+1) = \frac{D(D-2)(D+1)}{2}$ exactly matches with the number of the desired remaining constraints. Thus we expect to build $\pi^{aIJ} \propto n^{[I}E^{aJ]}$ on this new constraint surface, where $E^{aJ} := \sqrt{h}h^{ab}e^I_b$, h^{ab} is the inverse of $h_{ab} \equiv e^I_a e_{bI}$, n^I is the internal vector orthogonal to e^I_a and uniquely determined (up to a sign) by e^I_a through

$$n_I := \frac{1}{D!}\frac{1}{\sqrt{det(h)}}\epsilon^{a_1...a_D}\epsilon_{IJ_1...J_D}e^{J_1}_{a_1}..e^{J_D}_{a_D}. \tag{3.4}$$

Note that one has $n_I n^I = 1$ for $SO(D+1)$ and $n_I n^I = -1$ for $SO(1,D)$. In the following, we will choose the compact gauge group $SO(D+1)$ and require that

$$\pi^{aIJ} := 2\sqrt{det(h)}h^{ab}n^{[I}e^{J]}_b = 2n^{[I}E^{|a|J]}, \tag{3.5}$$

on the constraint surface of "Simplicity Constraint". It should be noted that $\frac{D(D-2)(D+1)}{2} = 0$ for $D = 2$ and hence no simplicity constraint is needed under this case.

To get an explicit expression of the simplicity constraint, for any given unit internal vector n_I, we define $E^{aI} := -\pi^{aIJ}n_J$ and its corresponding quantities:

$$Q^{ab} := E^a_I E^b_J \eta^{IJ}, Q_{ac}Q^{cb} := \delta^b_a, E^J_a := Q_{ab}E^{bI}, \tag{3.6}$$

where η^{IJ} is the internal metric. Furthermore, we define the transversal projector:

$$\bar{\eta}^I_J[n] := \delta^I_J - n^I n_J. \tag{3.7}$$

Using E^J_a and $\bar{\eta}^I_J$, we can define the tracefree and transverse projector:

$$P^{aIJ}_{bKL}[E] := \delta^a_b\bar{\eta}^I_{[K}\bar{\eta}^J_{L]} - \frac{2}{D-1}E^{a[I}E_{b[K}\bar{\eta}^{J]}_{L]}. \tag{3.8}$$

Next we define

$$\bar{\pi}_T^{aIJ} := P^{aIJ}_{bKL}\pi^{bKL} = \bar{\pi}^{aIJ} + \frac{2}{D-1}\bar{\pi}^{[I}E^{|a|J]}. \tag{3.9}$$

Note that $\bar{\pi}_T^{aIJ}$ satisfies $E_{aI}\bar{\pi}_T^{aIJ} = 0$ and $\bar{\pi}_T^{aIJ}n_I = 0$. The key observation is that $\bar{\pi}_T^{aIJ}$ has only $\frac{D(D-2)(D+1)}{2}$ degrees of freedom which is just the number of degrees of freedom we need to remove. Hence a given tensor π^{aIJ} can be decomposed into three parts:

$$\pi^{aIJ} = \bar{\pi}_T^{aIJ} - \frac{2}{D-1}\bar{\pi}^{[I}E^{|a|J]} + 2n^{[I}E^{|a|J]}, \tag{3.10}$$

where $\bar{\pi}^J := E_{aI}\bar{\pi}^{aIJ}$, and hence $\bar{\pi}^{[I}E^{|a|J]}$ is normal to n^I but not normal to E^{aI}. On the other hand, as shown in Ref.[28], one can always choose a suitable internal vector n_I such that

$$\bar{\pi}^J[\pi, n] = \bar{\pi}^{aIJ}[\pi, n]Q_{ab}[\pi, n]E^b_I = 0. \tag{3.11}$$

Thus, by employing the chosen n^I, One obtains an intrinsic decomposition: $\pi^{aIJ} = \bar{\pi}_T^{aIJ} + 2n^{[I}E^{|a|J]}$. Hence one would like to impose the simplicity constraint as the necessary and sufficient condition for a vanishing $\bar{\pi}_T^{aIJ}$. Let $D \geq 3$ and

$$S^{ab}_{\bar{M}} := \frac{1}{4}\epsilon_{I_0I_1I_2I_3\bar{M}}\pi^{aI_0I_1}\pi^{bI_2I_3}, \tag{3.12}$$

where $\bar{M}$ is any totally skew $(D-3)$-tuple of indices in $0, 1, .., D$, which stands for the set of the other $(D-3)$ antisymmetric indices $\{I_4, I_5, .., I_D\}$. Then for any unit vector n^I, one has [28]

$$S^{ab}_{\bar{M}} = 0, \forall\bar{M}, a, b \Leftrightarrow P^{aIJ}_{bKL}[\pi, n]\pi^{bKL} = 0. \tag{3.13}$$

Therefore the desired simplicity constraint reads $S^{ab}_{\bar{M}} = 0$.

Now we consider the Hamiltonian formalism of a $SO(D+1)$ gauge theory with connection A_{aIJ} and its conjugate momentum π^{bKL} as basic variables. These variables are subject to the Poisson brackets

$$\{A_{aIJ}(x), \pi^{bKL}(y)\} = 4\beta\delta^a_b\delta^K_{[I}\delta^L_{J]}\delta^D(x, y), \quad \{A_{aIJ}(x), A_{bKL}(y)\} = \{\pi^{aIJ}(x), \pi^{bKL}(y)\} = 0, \tag{3.14}$$

where β is the "Immirzi-like parameter" (it is structurally different from the Immirzi parameter in $D = 3$) in D dimensions. Then the Gaussian constraint and simplicity constraint read respectively [28]:

$$\begin{aligned} G^{IJ} &:= \mathcal{D}_a\pi^{aIJ} := \partial_a\pi^{aIJ} + 2A_a^{[I}{}_K\pi^{a|K|J]}, &(3.15)\\ S^{ab}_{\bar{M}} &:= \frac{1}{4}\epsilon_{IJKL\bar{M}}\pi^{aIJ}\pi^{bKL}. &(3.16)\end{aligned}$$

The ADM variables can be related to the Yang-Mills variables by the following map,

$$\begin{aligned} hh^{ab} &:= \frac{1}{2}\pi^{aIJ}\pi^b{}_{IJ}, &(3.17)\\ P^{ab} &:= \frac{1}{8\beta}\Big(h^{a[c}[A_{cIJ} - \Gamma_{cIJ}\pi^{b]IJ} + h^{b[c}[A_{cIJ} - \Gamma_{cIJ}]\pi^{a]IJ}\Big) \\ &=: \frac{1}{4}h^{d(a}K_{cIJ}\pi^{[b)IJ}\delta^{c]}_d, &(3.18)\end{aligned}$$

where $\Gamma_{cIJ}(\pi)$ satisfies ($\approx$ means vanishing on the simplicity constraint surface)

$$D_a\pi^{bIJ} := \partial_a\pi^{bIJ} + \Gamma^b_{ac}\pi^{cIJ} + 2\Gamma_a{}^{[I}{}_K\pi^{|bK|J]} - \Gamma^c_{ac}\pi^{bIJ} \approx 0. \tag{3.19}$$

Eq.(3.19) can be explicitly solved as

$$\Gamma_{aIJ}[\pi] := \frac{2}{D-1}T_{aIJ} + \frac{D-3}{D-1}\bar{T}_{aIJ} + \Gamma^b_{ac}T^c_{bIJ}, \tag{3.20}$$

where $T_{aIJ} := \pi_{bK[I}\partial_a\pi^{bK}{}_{J]}$, $\bar{T}_{aIJ} := \bar{\eta}^K_I\bar{\eta}^L_JT_{aKL}$, $T^c_{bIJ} := \pi_{bK[I}\pi^{cK}{}_{J]}$, and Γ^b_{ac} is the Levi-Civita connection compatible with h_{ab}. Using $\pi^{aIJ} \approx 2n^{[I}E^{|a|J]}$, one can show that $\Gamma_{aIJ}[\pi]$ is compatible with e^I_a, i.e.,

$$D_ae^I_b = \partial_ae^I_b - \Gamma^c_{ab}e^I_c + \Gamma_a^{IJ}e_{bJ} = 0. \tag{3.21}$$

on the simplicity constraint surface. It was shown in Ref. [28] that using the symplectic structure (3.14), one can correctly recover

$$\{h_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^D(x,y), \quad \{h_{ab}(x), h_{cd}(y)\} = \{P^{ab}(x), P^{cd}(y)\} = 0, \tag{3.22}$$

on the simplicity and Gaussian constraints surface. Hence, the map defined by Eqs. (3.17) and (3.18) gives a symplectic reduction from the Yang-Mills phase space to the ADM phase space. The diffeomorphism constraint (3.2) and Hamiltonian constraint (3.3) can be expressed in terms of the new variables as

$$\mathcal{V}_a = \frac{1}{2\beta}F_{abIJ}\pi^{bIJ}, \tag{3.23}$$

$$\begin{aligned}\mathcal{H} &= \frac{1}{2\sqrt{h}}\Big(F_{abIJ}\pi^{aIK}\pi^b{}_K{}^J + 4\bar{D}^{aIJ}_T(F^{-1})_{aIJ,bKL}\bar{D}^{bKL}_T + \frac{1}{(D-1)^2}[D_b{}^a D_a{}^b - (D_a{}^a)^2]\Big) \\ &+ \frac{1}{8\beta^2(D-1)^2\sqrt{h}}[D_b{}^a D_a{}^b - (D_a{}^a)^2],\end{aligned} \tag{3.24}$$

where $F_{abIJ} \equiv 2\partial_{[a}A_{b]IJ} + 2A_{a[I|K|}A_{|b|}{}^K{}_{J]}$ is the curvature of A_{aIJ}. Here we defined

$$D_b{}^a := \pi^{aK}{}_J(D_b\pi^{cJL})\pi_{cKL}, \tag{3.25}$$

$$(F^{-1})_{aIJ,bKL} := \frac{1}{4}[Q_{ab}\bar{\eta}_{K[I}\bar{\eta}_{J]L} - 2E_{b[I}\bar{\eta}_{J][K}E_{aL]}], \tag{3.26}$$

$$D^{aIJ} := \pi^{b[I}{}_K\mathcal{D}_b\pi^{a|K|J]}, \tag{3.27}$$

where $\mathcal{D}_a$ is the covariant differential of A acting only on internal indices, i.e.,

$$\mathcal{D}_a\pi^{bIJ} := \partial_a\pi^{bIJ} + 2A_a^{[I}{}_K\pi^{b|K|J]}, \tag{3.28}$$

and $\bar{D}^{aIJ}_T$ is the tracefree and transverse part of D^{aIJ} defined by

$$\bar{D}^{aIJ}_T \ : \ = P^{aIJ}_{bKL}\cdot D^{bKL}. \tag{3.29}$$

All of the constraints (3.15),(3.16),(3.23),(3.24) are proved to be of first-class [28].

B. Connection dynamics for STT in $D+1$ dimensions

It was recently shown in Ref. [25] that the STT in 3+1 dimensions can be cast into connection dynamical formalism. However, a connection-dynamical formalism for STT in arbitrary dimensions is still lacking. Thus our task now is to extend the approach introduced in the last subsection to formulate a connection dynamics of GR to $(D+1)$-dimensional STT. Recall that in order to build the connection dynamics of $(D+1)$-dimensional GR, we need to define the suitable canonical variables π^{aIJ} and A_{aIJ} of Yang-Mills fields and then construct the ADM phase space by symplectic reduction. For STT, the question becomes how to get the ADM-like phase space obtained in section II by a suitable symplectic reduction of a $so(D+1)$ Yang-Mills phase space. Note that, besides Yang-Mills variables, we also need a scalar field and its momentum. Hence the phase space of the gauge theory consists of the canonical pairs $(\tilde{A}_{aIJ}, \pi^{aIJ})$ and (ϕ, π), with basic Poisson brackets

$$\begin{aligned}\{\tilde{A}_{aIJ}(x), \pi^{bKL}(y)\} &= 4\beta\delta^a_b\delta^K_{[I}\delta^L_{J]}\delta^D(x,y), \quad \{\tilde{A}_{aIJ}(x), \tilde{A}_{bKL}(y)\} = \{\pi^{aIJ}(x), \pi^{bKL}(y)\} = 0, \\ \{\phi(x), \pi(y)\} &= \delta^D(x,y), \qquad\qquad\qquad\quad \{\phi(x), \phi(y)\} = \{\pi(x), \pi(y)\} = 0.\end{aligned} \tag{3.30}$$

To construct the ADM variables from the Yang-Mills variables, we first define

$$\beta\tilde{K}_{aIJ} = \tilde{A}_{aIJ} - \tilde{\Gamma}_{aIJ}, \tag{3.31}$$

where β is an arbitrary real number, and

$$\tilde{\Gamma}_{aIJ} := \Gamma_{aIJ}[\pi; x) + S_{aIJ}[\pi; x), \tag{3.32}$$

where S_{aIJ} refers to certain function vanishing on the simplicity constraint surface, and Γ_{aIJ} is defined by Eq.(3.20). As shown in [28], $\tilde{\Gamma}_{aIJ}$ can be chosen as the functional derivative of a generating function $F[\pi]$ such that K_{aIJ}

commutes with itself in Poisson brackets. This property will simplify the calculations of our constraint algebra. Then we define a map from the phase space of the gauge field coupled with the scalar field to ADM-like phase space of STT by

$$
\begin{aligned}
hh^{ab} &:= \frac{1}{2}\pi^{aIJ}\pi^{b}{}_{IJ}, && (3.33)\\
p^{ab} &:= \frac{1}{4}h^{d(a}\tilde{K}_{cIJ}\pi^{[b)IJ}\delta^{c]}_{d}, && (3.34)\\
\phi &:= \phi, && (3.35)\\
\pi &:= \pi. && (3.36)
\end{aligned}
$$

Note that the Gaussian constraint of the gauge theory reads

$$
G^{IJ} := \tilde{\mathcal{D}}_a\pi^{aIJ} = \partial_a\pi^{aIJ} + 2\tilde{A}_a^{[I}{}_K\pi^{a|K|J]}, \tag{3.37}
$$

while the simplicity constraint keep the same form as Eq.(3.16). Now it is straightforward to check that $h_{ab}[\pi]$ and $p^{ab}[A,\pi]$ defined in Eqs.(3.33) and (3.34) are Dirac observables with respect to the Gaussian and simplicity constraints and obey the standard Poisson brackets :

$$
\{h_{ab}(x), p^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^D(x,y), \quad \{h_{ab}(x), h_{cd}(y)\} = \{p^{ab}(x), p^{cd}(y)\} = 0. \tag{3.38}
$$

Therefore the map defined by Eqs. (3.33),(3.34),(3.35) and (3.36) is also a symplectic reduction.

To reformulate the geometrical dynamics of $(D+1)$-dimensional STT by connection dynamics. We first consider the sector of $\omega(\phi) \neq -\frac{D}{D-1}$. Eq.(3.34) implies that $\tilde{K}_a{}^b := \frac{1}{4}\tilde{K}_{aIJ}\pi^{bIJ}$ is related to the extrinsic curvature in Eq.(2.17) by In Eq.(2.5), we define

$$
\begin{aligned}
\tilde{K}_a{}^b &= \phi\sqrt{h}K_a{}^b + \frac{\sqrt{h}\delta^b_a}{(D-1)N}(\dot{\phi} - N^c\partial_c\phi)\\
&= \phi\sqrt{h}K_a{}^b + \frac{2(\frac{D-1}{2}\phi\pi - p)\delta^b_a}{(D-1)(D+(D-1)\omega)}. && (3.39)
\end{aligned}
$$

Straightforward calculations show that the original diffeomorphism and Hamiltonian constraints (2.9) and (2.10) can be respectively written in terms of new variables as

$$
\begin{aligned}
\mathcal{H}_a &= \frac{1}{2\beta}\tilde{F}_{abIJ}\pi^{bIJ} + \pi\partial_a\phi, && (3.40)\\
\mathcal{H} &= \frac{\phi}{2\sqrt{h}}\Big(\tilde{F}_{abIJ}\pi^{aIK}\pi^b{}_K{}^J + 4\tilde{\bar{D}}^{aIJ}_T(\tilde{F}^{-1})_{aIJ,bKL}\tilde{\bar{D}}^{bKL}_T + \frac{1}{(D-1)^2}[\tilde{D}_b{}^a\tilde{D}_a{}^b - (\tilde{D}_a{}^a)^2]\Big)\\
&+ \frac{1}{8\phi\beta^2(D-1)^2\sqrt{h}}[\tilde{D}_b{}^a\tilde{D}_a{}^b - (\tilde{D}_a{}^a)^2] + \frac{(\tilde{D}_a{}^a - 2\beta(D-1)\phi\pi)^2}{8\phi\beta^2(D-1)(D+(D-1)\omega)\sqrt{h}}\\
&+ \frac{1}{2}\sqrt{h}\left(\frac{\omega(\phi)}{\phi}(\tilde{D}_a\phi)\tilde{D}^a\phi + 2\tilde{D}_a\tilde{D}^a\phi + 2\xi(\phi)\right), && (3.41)
\end{aligned}
$$

where $F_{abIJ} \equiv 2\partial_{[a}A_{b]IJ} + 2A_{a[I|K|}A_{|b|}{}^K{}_{J]}$ and the definitions for $\tilde{D}_a{}^b$,$(\tilde{F}^{-1})_{aIJ,bKL}$ and $\tilde{\bar{D}}^{aIJ}_T$ take the same forms as Eqs.(3.25),(3.26) and (3.29) except for A_{aIJ} being replaced by $\tilde{A}_{aIJ}$, and the generalized derivative $\tilde{D}_a$ satisfies $\tilde{D}_a\pi^{bIJ} := \partial_a\pi^{bIJ} + \Gamma^b_{ac}\pi^{cIJ} + 2\tilde{\Gamma}_a{}^{[I}{}_K\pi^{|bK|J]} - \Gamma^c_{ac}\pi^{bIJ} = 0$ on the simplicity constraint surface. The total Hamiltonian can now be expressed as a linear combination

$$
H_{total} = \int_\Sigma d^Dx(c^{\bar{M}}_{ab}S^{ab}_{\bar{M}} + \frac{1}{2}f_{IJ}G^{IJ} + N^a\mathcal{H}_a + N\mathcal{H}). \tag{3.42}
$$

It is easy to check that the smeared Gaussian constraint $G[f] := \int_\Sigma d^Dx\frac{1}{2}f_{IJ}(x)G^{IJ}(x)$ generates $SO(D+1)$ gauge transformations on the phase space as

$$
\Big\{\tilde{A}_{aIJ}, G[f]\Big\} = -2\beta\tilde{\mathcal{D}}_af_{IJ}, \quad \Big\{\pi^{aIJ}, G[f]\Big\} = 2\beta[f,\pi^a]^{IJ}. \tag{3.43}
$$

The smeared diffeomorphism constraint $\tilde{\mathcal{H}}[\overrightarrow{N}] := \int_\Sigma d^D x N^a(\mathcal{H}_a - \frac{1}{2\beta}\tilde{A}_{aIJ}G^{IJ})$ generates the spatial diffeomorphism transformations on the phase space as

$$\Big\{\tilde{A}_{aIJ}, \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} \;=\; 2\mathcal{L}_{\vec{N}}\tilde{A}_{aIJ}, \quad \Big\{\pi^{aIJ}, \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} = 2\mathcal{L}_{\vec{N}}\pi^{aIJ}, \tag{3.44}$$

$$\Big\{\phi, \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} \;=\; \mathcal{L}_{\vec{N}}\phi, \qquad \Big\{\pi, \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} = \mathcal{L}_{\vec{N}}\pi. \tag{3.45}$$

Thus we can show that the constraint algebra has the following Poisson subalgebra:

$$\Big\{G[f], S^{ab}_{\bar{M}}[d^{\bar{M}}_{ab}]\Big\} \;=\; S^{ab}_{\bar{M}}[\sum_{i=1}^{D-3} 2\beta f^{I_i}{}_{I'_i} d_{ab}^{I_1\ldots I'_i\ldots I_{D-3}}], \tag{3.46}$$

$$\Big\{S^{ab}_{\bar{M}}[c^{\bar{M}}_{ab}], S^{ab}_{\bar{M}}[d^{\bar{M}}_{ab}]\Big\} \;=\; 0, \tag{3.47}$$

$$\Big\{G[f], G[g]\Big\} \;=\; -2\beta G[f,g], \tag{3.48}$$

$$\Big\{G[f], \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} \;=\; G[-\mathcal{L}_{\vec{N}} f], \tag{3.49}$$

$$\Big\{S^{ab}_{\bar{M}}[c^{\bar{M}}_{ab}], \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} \;=\; 2S^{ab}_{\bar{M}}[\mathcal{L}_{\overrightarrow{N}} c^{\bar{M}}_{ab}], \tag{3.50}$$

$$\Big\{\tilde{\mathcal{H}}[\overrightarrow{M}], \tilde{\mathcal{H}}[\overrightarrow{N}]\Big\} \;=\; 2\tilde{\mathcal{H}}([\overrightarrow{M}, \overrightarrow{N}]). \tag{3.51}$$

The constraint brackets between the smeared Hamiltonian constraint with itself and other constraints are also closed as

$$\Big\{G[f], \tilde{\mathcal{H}}[M]\Big\} \;=\; 0, \tag{3.52}$$

$$\Big\{\tilde{\mathcal{H}}[\overrightarrow{N}], \tilde{\mathcal{H}}[M]\Big\} \;=\; \tilde{\mathcal{H}}[L_{\overrightarrow{N}} M], \tag{3.53}$$

$$\Big\{S^{ab}_{\bar{M}}[c^{\bar{M}}_{ab}], \tilde{\mathcal{H}}[M]\Big\} \;=\; 0, \tag{3.54}$$

$$\Big\{\tilde{\mathcal{H}}[M], \tilde{\mathcal{H}}[N]\Big\} \;=\; \int_\Sigma \frac{1}{2h}\pi^{aIK}\pi^b{}_K{}^J(ND_bM - MD_bN)\mathcal{H}_a + \frac{[\pi^a D_a N, \pi^b D_b M]^{IJ}}{4h\beta} G_{IJ}. \tag{3.55}$$

The detailed calculation of Eq.(3.55) will be presented in appendix A. Hence all the constraints now are of first class. Thus the STT of gravity in the sector $\omega(\phi) \neq \frac{D}{D-1}$ have been cast into the $so(D+1)$-connection dynamics formalism with a first-class contraint system.

Next we consider the other sector of $\omega(\phi) = -\frac{D}{D-1}$. Besides the diffeomorphism and Hamiltonian constraints, the geometrical dynamics of STT contains an extra primary conformal constraint (2.20). On the phase space of the gauge field coupled with the scalar field, the total Hamiltonian can be expressed as a liner combination

$$H_{total} = \int_\Sigma d^D x (c^{\bar{M}}_{ab} S^{ab}_{\bar{M}} + \frac{1}{2} f_{IJ} G^{IJ} + N^a \mathcal{H}_a + N\mathcal{H} + \lambda C), \tag{3.56}$$

where the simplicity, Gaussian and diffeomorphism constraints keep the same form as Eqs.(3.16),(3.37) and (3.40), while the Hamiltonian and conformal constraints read respectively:

$$\begin{aligned}\mathcal{H} \;&=\; \frac{\phi}{2\sqrt{h}}\Big(\tilde{F}_{abIJ}\pi^{aIK}\pi^b{}_K{}^J + 4\tilde{\bar{D}}^{aIJ}_T (F^{-1})_{aIJ,bKL}\tilde{\bar{D}}^{bKL}_T + \frac{1}{(D-1)^2}[\tilde{D}_b{}^a \tilde{D}_a{}^b - (\tilde{D}_a{}^a)^2]\Big) \\ &+\; \frac{1}{8\phi\beta^2(D-1)^2\sqrt{h}}[\tilde{D}_b{}^a\tilde{D}_a{}^b - (\tilde{D}_a{}^a)^2] + \frac{1}{2}\sqrt{h}\left(\frac{-D}{(D-1)\phi}(\tilde{D}_a\phi)\tilde{D}^a\phi + 2\tilde{D}_a\tilde{D}^a\phi + 2\xi(\phi)\right),\end{aligned} \tag{3.57}$$

$$S \;=\; \frac{1-D}{2}(\tilde{K}_a{}^a + \pi\phi). \tag{3.58}$$

After solving the second-class constraint as shown in section II, straightforward calculations show that the constraint

algebra is still closed as:

$$\{S^{ab}_{\bar{M}}[d^{\bar{M}}_{ab}], C[\lambda]\} = S^{ab}_{\bar{M}}[(D-1)d^{\bar{M}}_{ab}], \tag{3.59}$$

$$\{G[f], C[\lambda]\} = 0, \tag{3.60}$$

$$\{C[\lambda], \tilde{\mathcal{H}}[M]\} = \tilde{\mathcal{H}}[\frac{\lambda M}{2}], \tag{3.61}$$

$$\begin{aligned}\{\tilde{\mathcal{H}}[N], \tilde{\mathcal{H}}[M]\} &= \int_\Sigma \frac{1}{2h}\pi^{aIK}\pi^b{}_K{}^J(ND_bM - MD_bN)\mathcal{H}_a + \frac{[\pi^a D_a N, \pi^b D_b M]^{IJ}}{4h\beta}G_{IJ} \\ &+ C[\frac{2D_a\phi}{(D-1)\phi}(ND^aM - MD^aN)].\end{aligned} \tag{3.62}$$

The derivation of Eq.(3.59) will be given in appendix A. Obviously the Poisson brackets among the other constraints are also weakly equal to zero. Hence all the constraints in this case are also of first class. To summarize, the STT of gravity in both sectors of $\omega(\phi) \neq -\frac{D}{D-1}$ and $\omega(\phi) = -\frac{D}{D-1}$ have been cast into the $so(D+1)$-connection dynamical formalism with first-class constraint system.

IV. CONCLUDING REMARKS

As candidate modified gravity theories, STT have received increased attention in issues of "dark Universe" and nontrivial tests on gravity beyond GR. On the other hand, modern theoretical research explores the possibility of higher dimensional spacetime. In order to study the non-perturbative quantization of higher dimensional STT in LQG scheme, it is necessary to build the connection dynamics of STT in higher spacetime dimensions. The achievements in this paper are the derivation of the detailed Hamiltonian structure of STT and the construction of their connection dynamics in $(D+1)$-dimensional spacetime. First, by doing Hamiltonian analysis, we derive the Hamiltonian formulation of STT from the $(D+1)$-dimensional Lagrangian formulation in ADM-like variables. Two sectors are marked off by the coupling parameter $\omega(\phi)$. In the sector of $\omega(\phi) \neq -\frac{D}{D-1}$, the canonical structure and constraint algebra of STT are similar to those of GR coupled with a scalar field. In the other sector of $\omega(\phi) = -\frac{D}{D-1}$, the feasible theories are restricted and a new primary constraint generating conformal transformations of spacetime is obtained. The canonical structure and constraint algebra are also obtained. All the Hamiltonian structures are direct generalization of 4-dimensional case. Next we successfully construct a $so(D+1)$ Hamiltonian connection formulation of STT in $D+1$ spacetime dimensions, from which the ADM-like Hamiltonian formulation can be obtained by a sympletic reduction. As in higher dimensional GR, a simplicity constraint has to be introduced into the higher dimensional connection dynamics of STT for the symplectic reduction. Finally, we show that the constraint algebra in both sectors of STT are also closed in the connection dynamical formalism.

It should be noted that we have casted $(D+1)$-dimensional STT into the connection dynamical formalism with the compact $SO(D+1)$ structure group. Hence it is straightforward to employ the techniques of LQG and those developed in Refs.[25, 31] to quantize the higher dimensional STT non-perturbatively. This opens the possibility to confront the effects of non-perturbative LQG with those of higher dimensional quantum gravity theories such as string/M theory.

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Appendix A

In this appendix, we will present the detailed calculation for the Poisson brackets (3.55) and (3.59). Since the simplicty constraint commutes with itself as well as Gaussian and diffeomorphism constraints. We can write the smeared Hamiltonian constraint modulo the simplicity constraint as

$$\begin{aligned}\tilde{\mathcal{H}}[M] &= \int_\Sigma d^D x M\Big[\frac{1}{2\sqrt{h}\phi}(\tilde{K'}_{aIJ}\pi^{bIJ}\tilde{K'}_{bKL}\pi^{aKL} - \tilde{K'}_{aIJ}\pi^{aIJ}\tilde{K'}_{bKL}\pi^{bKL}) - \frac{1}{2}\phi\sqrt{h}R^{(D)} + \sqrt{h}\xi(\phi) \\ &+ \frac{1}{2(\frac{D}{D-1}+\omega)\phi\sqrt{h}}(\tilde{K'}_{aIJ}\pi^{aIJ} + \pi\phi)^2 + \frac{\omega}{2\phi}\sqrt{h}D_a\phi D^a\phi + \sqrt{h}D_aD^a\phi\Big],\end{aligned} \tag{4.1}$$

where $\tilde{K}'_{aIJ} := \frac{1}{4}\tilde{K}_{aIJ}$. To calculate the Poisson bracket between two smeared Hamiltonian constraints, we note that the non-vanishing contributions come only form the terms which contain the derivative of canonical variables. Those terms are $\int_\Sigma \frac{N\sqrt{h}\omega}{2\phi}(D_a\phi)D^a\phi$, which only contains the derivative of ϕ, and $\int_\Sigma d^Dx N\sqrt{h}D_aD^a\phi$, which contains both the derivative of π^{aIJ} and the derivative of ϕ, and $\int_\Sigma -\frac{1}{2}\phi N\sqrt{h}R^{(D)}$, which only contain the derivative of π^{aIJ}. Hence we first use $\{\phi(x),\pi(y)\} = \delta^D(x,y)$ to calculate

$$\begin{aligned}
&\{\int_\Sigma N\sqrt{h}D_aD^a\phi, \int_\Sigma \frac{M}{2(\frac{D}{D-1}+\omega)\phi\sqrt{h}}(\tilde{K}'_{aIJ}\pi^{aIJ}+\pi\phi)^2\}_{(\phi,\pi)} - M \leftrightarrow N \\
=\;& \frac{1}{\frac{D}{D-1}+\omega}\int_\Sigma (ND^aM - MD^aN)D_a(\pi\phi + \tilde{K}'_{bIJ}\pi^{bIJ}),
\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
&\{\int_\Sigma \frac{N\sqrt{h}\omega}{2\phi}(D_a\phi)D^a\phi, \int_\Sigma \frac{M}{2(\frac{D}{D-1}+\omega)\phi\sqrt{h}}(\tilde{K}'_{bIJ}\pi^{bIJ}+\pi\phi)^2\}_{(\phi,\pi)} - M \leftrightarrow N \\
=\;& \frac{\omega}{\frac{D}{D-1}+\omega}\int_\Sigma (ND^aM - MD^aN)(\pi\phi + \tilde{K}'_{bIJ}\pi^{bIJ})\frac{D_a\phi}{\phi}.
\end{aligned} \tag{4.3}$$

Note also that

$$N\sqrt{h}D_aD^a\phi = N\sqrt{h}h^{ab}(\partial_a\partial_b\phi - \Gamma^c_{ab}\partial_c\phi). \tag{4.4}$$

Since only Γ^c_{ab} contains the derivative of π^{aIJ}, we consider

$$\begin{aligned}
N\sqrt{h}h^{ab}\Gamma^c_{ab}\partial_c\phi &= \frac{N}{2}\sqrt{h}h^{ab}(\partial_c\phi)\Big(h^{cd}(-\partial_a h_{bd} - \partial_b h_{ad} + \partial_d h_{ab})\Big) \\
&= \frac{N}{2}\sqrt{h}(\partial_c\phi)\Big(2\partial_a(\frac{\pi^{aIJ}\pi^c{}_{IJ}}{2h}) - h_{ab}\partial^c(\frac{\pi^{aIJ}\pi^b{}_{IJ}}{2h})\Big).
\end{aligned} \tag{4.5}$$

Therefore, we use $\Big\{\tilde{K}'_{aIJ}(x), \pi^{bKL}(y)\Big\} = \delta^b_a\eta^{[K}_I\eta^{L]}_J\delta(x,y)$ to calculate

$$\begin{aligned}
&\Big\{\int_\Sigma N\sqrt{h}(\partial_c\phi)\partial_a(\frac{\pi^{aIJ}\pi^c{}_{IJ}}{2h}), \int_\Sigma \frac{M}{2\sqrt{h}}\Big(\frac{1}{\phi}(\tilde{K}'_{dMN}\pi^{bMN}\tilde{K}'_{bKL}\pi^{dKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K}'_{dMN}\pi^{dMN}\tilde{K}'_{bKL}\pi^{bKL}) \\
+\;& \frac{2}{(\frac{D}{D-1}+\omega)}\tilde{K}'_{dMN}\pi^{dMN}\pi\Big)\Big\}_{(\tilde{K}',\pi)} - M \leftrightarrow N \\
=\;& \int_\Sigma \frac{1}{4}M(\partial_a N)(D_c\phi)\frac{2\pi^c{}_{IJ}}{h}\Big(\frac{2}{\phi}(\pi^{bIJ}\tilde{K}'_{bKL}\pi^{aKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\pi^{aIJ}\tilde{K}'_{bKL}\pi^{bKL}) + \frac{2}{(\frac{D}{D-1}+\omega)}\pi^{aIJ}\pi\Big) \\
+\;& \frac{1}{4}M(\partial_a N)(D_c\phi)\frac{\pi^{aIJ}\pi^c{}_{IJ}}{h}(\frac{-1}{D-1}\pi_{dKL})\Big(\frac{2}{\phi}(\pi^{bKL}\tilde{K}'_{bMN}\pi^{dMN} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\pi^{dKL}\tilde{K}'_{bMN}\pi^{bMN}) \\
+\;& \frac{2}{(\frac{D}{D-1}+\omega)}\pi^{dKL}\pi\Big) - M \leftrightarrow N
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\Big\{\int_\Sigma -\frac{N}{4}\sqrt{h}(\partial_c\phi)h_{ae}\partial^c(\frac{\pi^{aIJ}\pi^{eIJ}}{h}), \int_\Sigma \frac{M}{2\sqrt{h}}\Big(\frac{1}{\phi}(\tilde{K}'_{dMN}\pi^{bMN}\tilde{K}'_{bKL}\pi^{dKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K}'_{dMN}\pi^{dMN}\tilde{K}'_{bKL}\pi^{bKL}) \\
+\;& \frac{2}{(\frac{D}{D-1}+\omega)}\tilde{K}'_{dMN}\pi^{dMN}\pi\Big)\Big\}_{(\tilde{K}',\pi)} - M \leftrightarrow N \\
=\;& \int_\Sigma -\frac{1}{8}M\partial^c N D_c\phi h_{ae}\frac{2\pi^{eIJ}}{h}\Big(\frac{2}{\phi}(\pi^{bIJ}\tilde{K}'_{bKL}\pi^{aKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\pi^{aIJ}\tilde{K}'_{bKL}\pi^{bKL}) + \frac{2}{(\frac{D}{D-1}+\omega)}\pi^{aIJ}\pi\Big) \\
-\;& \frac{1}{8}M\partial_a N D_c\phi\frac{\pi^{aIJ}\pi^{cIJ}}{2h}(\frac{-2D}{D-1}\pi_{dKL})\Big(\frac{2}{\phi}(\pi^{bKL}\tilde{K}'_{bMN}\pi^{dMN} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\pi^{dKL}\tilde{K}'_{bMN}\pi^{bMN}) + \frac{2}{(\frac{D}{D-1}+\omega)}\pi^{dKL}\pi\Big) \\
-\;& M \leftrightarrow N.
\end{aligned} \tag{4.7}$$

The combination of above two Poisson brackets equals to

$$\int_\Sigma (ND^aM - MD^aN)\Big(- \frac{\frac{1}{D-1}}{\frac{D}{D-1}+\omega}\pi D_a\phi - \frac{2}{\phi}(\tilde{K'}_{bKL}\pi^{cKL}h_{ac}D^b\phi - \frac{\frac{2}{D-1}+\omega}{2(\frac{D}{D-1}+\omega)}\tilde{K'}_{bKL}\pi^{bKL}D_a\phi)\Big). \tag{4.8}$$

The variation of the terms containing a derivative in $\int_\Sigma -\frac{1}{2}\phi N\sqrt{h}R^{(D)}$ reads

$$\begin{aligned} &\int_\Sigma \frac{1}{2}\sqrt{h}(-D^aD^b(\phi N) + h^{ab}D_cD^c(\phi N))\delta h_{ab} \\ = &\int_\Sigma \frac{1}{2}\sqrt{h}(D_aD_b(\phi N) - h_{ab}D_cD^c(\phi N))\delta(\frac{\pi^{aIJ}\pi^b{}_{IJ}}{2h}). \end{aligned} \tag{4.9}$$

Thus we have

$$\begin{aligned} &\Big\{ \int_\Sigma -\frac{1}{2}\phi N\sqrt{h}R^{(D)}, \int_\Sigma \frac{M}{2\sqrt{h}}\Big(\frac{1}{\phi}(\tilde{K'}_{dMN}\pi^{eMN}\tilde{K'}_{eKL}\pi^{dKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K'}_{dKL}\pi^{dKL}\tilde{K'}^{m}_{e}\pi^{eMN}) \\ + &\frac{2}{(\frac{D}{D-1}+\omega)}\tilde{K'}_{dMN}\pi^{dMN}\pi\Big)\Big\} - M\leftrightarrow N \\ = &\int_\Sigma -\frac{1}{8}(MD_aD_b(\phi N) - h_{ab}MD_cD^c(\phi N))\frac{2\pi^b{}_{IJ}}{h}\Big(\frac{2}{\phi}(\pi^{eIJ}\tilde{K'}_{eKL}\pi^{aKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K'}_{dKL}\pi^{dKL}\pi^{aIJ}) \\ + &\frac{2}{(\frac{D}{D-1}+\omega)}\pi^{aIJ}\pi\Big) - \frac{1}{8}(-(D-1)MD_cD^c(\phi N))(\frac{-2}{D-1}\pi_{aIJ})\Big(\frac{2}{\phi}(\pi^{eIJ}\tilde{K'}_{eKL}\pi^{aKL} - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K'}_{dKL}\pi^{dKL}\pi^{aIJ}) \\ + &\frac{2}{(\frac{D}{D-1}+\omega)}\pi^{aIJ}\pi\Big) - M\leftrightarrow N \\ \\ = &\int_\Sigma (ND_cD^cM - MD_cD^cN)\Big(\frac{1}{\phi(\frac{D}{D-1}+\omega)}\phi\pi - \frac{\frac{1}{D-1}+\omega}{\frac{D}{D-1}+\omega}\tilde{K'}_{aKL}\pi^{aKL}\Big) + (ND_cM - MD_cN)D^c\phi\Big(\frac{2}{(\frac{D}{D-1}+\omega)}\pi \\ - &\frac{\frac{2}{D-1}+2\omega}{\phi(\frac{D}{D-1}+\omega)}\tilde{K'}_{aKL}\pi^{aKL}\Big) + (ND_aD^bM - MD_aD^bN)\tilde{K'}_{bKL}\pi^{aKL} + (ND_aM - MD_aN)\frac{2D^b\phi}{\phi}\tilde{K'}_{bKL}\pi^{aKL}. \end{aligned} \tag{4.10}$$

Taking account of Eqs. (4.2)-(4.10), we obtain

$$\begin{aligned} &\{\tilde{\mathcal{H}}(N), \tilde{\mathcal{H}}(M)\} \\ = &\int_\Sigma (ND_cD^cM - MD_cD^cN)(-\tilde{K'}_{aKL}\pi^{aKL}) + (ND^aM - MD^aN)(\pi D_a\phi) + (ND_aD^bM - MD_aD^bN)\tilde{K'}_{bKL}\pi^{aKL} \\ = &\int_\Sigma (ND^aM - MD^aN)V_a + (D_aMD_bN - D_bMD_aN)\tilde{K}^{[ab]} \\ = &\int_\Sigma \frac{1}{2h}\pi^{aIK}\pi^b{}_K{}^J(ND_bM - MD_bN)V_a + ((D_aM)D_bN)2\tilde{K}^{[ab]}. \end{aligned} \tag{4.11}$$

Note that $\tilde{K}^{[ab]}$ is constrained to vanish by the Gaussian and simplicity constraint. To see this, we consider

$$\begin{aligned} G_{IJ} :&= \mathcal{D}_a\pi^a{}_{IJ} = \partial_a\pi^a{}_{IJ} + 2\tilde{A}_{aK[I}\pi^a{}_{J]}{}^K \\ &\approx -2\beta\tilde{K}_{a[I}E^a_{J]} + 2\beta\tilde{K}_{[I}n_{J]} =: \bar{G}_{IJ} + 2n_{[I}G_{J]}, \end{aligned} \tag{4.12}$$

where $\tilde{K}_{aI} := -\tilde{K}_{aLI}n^L$ and $\tilde{K}_I := K_{aLI}E^{aL}$. It follows that $\tilde{\bar{K}}_I = 0$ and $\tilde{K}_{a[I}E^a_{J]} = 0$ on the Gaussian constraint surface. Hence we have

$$\tilde{K}^{[ab]}E_{aI}E_{bJ} \approx \frac{1}{2}h^{c[a}\tilde{K}_{cL}E^{b]L}E_{aI}E_{bJ} = \frac{1}{2h}\tilde{K}_{a[I}E^a_{J]}. \tag{4.13}$$

Therefore we have $\tilde{K}^{[ab]} = \frac{1}{4h\beta}\bar{G}_{IJ}E^{aI}E^{bJ} \approx -\frac{1}{4h\beta}G_{IJ}\pi^{aIK}\pi^{b}{}_{K}{}^{J}$ on the simplicity constraint surface. Hence the Poisson bracket (3.55) can be obtained by Eq.(4.11).

Next we calculate the Poisson bracket (3.59). We notice that the non-vanishing contribution in the conformal constraint coming only from the first term $\tilde{K}^a_a$. Hence we have

$$\begin{aligned}
\{S^{bc}_{\bar{M}}[d^{\bar{M}}_{bc}], \frac{1-D}{2}\tilde{K}^a_a(y)\} &= \int_\Sigma d^D x \frac{1-D}{2} d^{\bar{M}}_{bc}(x)\{S^{bc}_{\bar{M}}(x), \tilde{K}^a_a(y)\} \\
&= \frac{D-1}{8\beta}\int_\Sigma d^D x d^{\bar{M}}_{bc}(x)\epsilon_{ABCD\bar{M}}\pi^{aIJ}\pi^{bAB}\{\pi^{cCD}, A_{aIJ}\} \\
&= S^{ab}_{\bar{M}}[(D-1)d^{\bar{M}}_{ab}].
\end{aligned} \tag{4.14}$$

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